## Chapter 1

## The Fourier Transform and its Inverse

### 1.1 The Fourier Transform

Fourier analysis is concerned with the mathematics associated with a particular type of integral. This integral can be written in the form

$$
\begin{equation*}
G(f)=\int_{-\infty}^{\infty} g(t) e^{-j 2 \pi f t} d t \tag{1.1}
\end{equation*}
$$

where $G(f)$ is said to be the Fourier transform of the function $g(t)$. If $t$ has the dimensions of time, then $g(t)$ can be thought of as a time signal. The dimensions of $f$ are then inverse time with units such as cycles/sec or Hertz $(\mathrm{Hz})$. For time signals the Fourier transform is often written in terms of the radian frequency $\omega=2 \pi f$. In this case Eq. (1.1) can be written as

$$
\begin{equation*}
G(\omega)=\int_{-\infty}^{\infty} g(t) e^{-j \omega t} d t \tag{1.2}
\end{equation*}
$$

The property of Fourier transforms that makes them useful is the fact that an inverse relation exists between $G(f)$ and $g(t)$. Thus, if $G(f)$ is given in terms of $g(t)$ by Eq. (1.1), it will be shown that $g(t)$ is given in terms $G(f)$ by the inverse relation

$$
\begin{equation*}
g(t)=\int_{-\infty}^{\infty} G(f) e^{j 2 \pi f t} d f \tag{1.3}
\end{equation*}
$$

The inverse of Eq. (1.2) is given by

$$
\begin{equation*}
g(t)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} G(\omega) e^{j \omega t} d \omega \tag{1.4}
\end{equation*}
$$

which follows from Eq. (1.3) with the change of variables $\omega=2 \pi f$. Sometimes in dealing with time signals the factor $1 / 2 \pi$ in Eq. (1.4) is replaced by the factor $1 / \sqrt{2 \pi}$ multiplying the integrals in both Eqs. (1.2) and (1.4). In using the forms of Eqs. (1.1) and (1.3), the factor $2 \pi$ appears in the exponents and does not appear in front of either integral. We will consistently use the forms given in Eqs. (1.1) and (1.3) for the Fourier transform and its inverse.

In this chapter, we will explore the nature of the relationship between Eqs. (1.1) and (1.3). We begin by calculating the Fourier transform of a specific function $g(t)$ in Example 1.

## Example 1: The rect Function

Let $g(t)$ be a rectangular pulse of width 1 and height 1 as shown in Fig. 1.1 and defined by

$$
g(t)=\operatorname{rect}(t)=\left\{\begin{array}{rr}
1 & |t| \leq 1 / 2 \\
0 & \text { otherwise }
\end{array}\right.
$$



Figure 1.1 The function $\operatorname{rect}(t)$

From Eq. (1.1), we can then calculate $G(f)$ as

$$
\begin{aligned}
G(f) & =\int_{-1 / 2}^{1 / 2} e^{-j 2 \pi f t} d t=\left.\frac{e^{-j 2 \pi f t}}{-j 2 \pi f}\right|_{-1 / 2} ^{1 / 2} \\
& =\frac{e^{-j \pi f}-e^{j \pi f}}{-j 2 \pi f}=\frac{\sin \pi f}{\pi f}
\end{aligned}
$$

The sinc function is defined by

$$
\begin{equation*}
\operatorname{sinc} f=\frac{\sin \pi f}{\pi f} \tag{1.5}
\end{equation*}
$$

and is shown in Fig. 1.2. Thus, $G(f)=\operatorname{sinc} f$ and we can write

$$
\begin{equation*}
\operatorname{rect}(t) \Leftrightarrow \operatorname{sinc}(f) \tag{1.6}
\end{equation*}
$$

where the symbol $\Leftrightarrow$ can be read "has as its Fourier transform."


Figure 1.2 The function $\operatorname{sinc}(f)$

## Example 2: The delta Function

As a second example, calculate $G(f)$ when $g(t)$ is a rectangular pulse of width $N$ as shown in Fig. 1.3. We can write $g(t)=\operatorname{rect}(t / N)$. From Eq. (1.1)


Figure 1.3 The function $\operatorname{rect}(t / N)$

$$
\begin{align*}
G(f) & =\int_{-N / 2}^{N / 2} e^{-j 2 \pi f t} d t=\left.\frac{e^{-j 2 \pi f t}}{-j 2 \pi f}\right|_{-N / 2} ^{N / 2} \\
& =\frac{\sin N \pi f}{\pi f}=N \operatorname{sinc}(N f) \tag{1.7}
\end{align*}
$$

Thus,

$$
\begin{equation*}
\operatorname{rect}\left(\frac{t}{N}\right) \Leftrightarrow N \operatorname{sinc}(N f) \tag{1.8}
\end{equation*}
$$

The function $N \operatorname{sinc}(N f)$ is shown in Fig. 1.4.


Figure 1.4 The function $N \operatorname{sinc}(N f)$

Consider what happens now if $N$ increases without limit. The function $\operatorname{rect}(t / N)$ approaches the constant 1 . The value of $N \operatorname{sinc}(N f)$ at $f=0$ grows without limit while at the same time the width of the function decreases and approaches zero. Thus, in the limit as $N \rightarrow \infty$ the function $N \operatorname{sinc}(N f)$ approaches something that looks like an infinite spike at the origin. This resembles a special kind of function called a delta function, written as $\delta(x)$, which can be defined in a number of different ways. One common way of defining the delta function is as follows

$$
\begin{align*}
& \delta(x)=0 \quad \text { if } x \neq 0 \\
& \int_{-\infty}^{\infty} \delta(x) d x=1 \tag{1.9}
\end{align*}
$$

That is, the delta function $\delta(x)$ is equal to zero everywhere except at $x=0$, but is such that the area under the delta function remains equal to unity. Can $N \operatorname{sinc}(N x)$ approximate a delta function for large values of $N$ ? We have already seen that in the limit $N \rightarrow \infty$ the width of $N \operatorname{sinc}(N x)$ goes to zero. We should therefore check to see if the area under the function $N \operatorname{sinc}(N x)$ is equal to one.

$$
\begin{aligned}
\int_{-\infty}^{\infty} N \operatorname{sinc}(N x) d x & =\int_{-\infty}^{\infty} N \frac{\sin N \pi x}{N \pi x} d x \\
& =\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin z}{z} d z=\frac{1}{\pi} \pi=1
\end{aligned}
$$

where the integral

$$
\int_{-\infty}^{\infty} \frac{\sin z}{z} d z=\pi
$$

can be obtained from integral tables or evaluated by certain techniques such as contour integration. Thus, we see that for large values of $N$ the function $N \operatorname{sinc}(N x)$ can approximate a delta function and should approach a delta function in the limit as $N \rightarrow \infty$.

Instead of defining the delta function by Eq. (1.9), an alternate approach is to define the delta function in terms of its sifting property,

$$
\begin{equation*}
g(0)=\int_{-\infty}^{\infty} \delta(z) g(z) d z \tag{1.10}
\end{equation*}
$$

or, more generally

$$
\begin{equation*}
g(t)=\int_{-\infty}^{\infty} \delta(z-t) g(z) d z \tag{1.11}
\end{equation*}
$$

Is this definition plausible and consistent with the definition given by Eq. (1.9)? Figure 1.5 a shows $\delta(z)$ multiplying an arbitrary function $g(z)$ as in the integrand of Eq. (1.10). Since $\delta(z)=0$ except at $z=0$, the value of the integrand will be zero except at $z=0$. If in a small region about $z=0$ we approximate $g(z)$ by its value at $z=0$, we can write

$$
\int_{-\infty}^{\infty} \delta(z) g(z) d z=g(0) \int_{-\infty}^{\infty} \delta(z) d z=g(0)
$$

A similar argument can be applied to Eq. (1.11) using Fig. 1.5b. Thus, Eqs. (1.10) and (1.11) seem plausible if not rigorously derivable from Eq. (1.9). Delta functions are useful because of this sifting property. If the delta function is defined directly in terms of the sifting property of Eq. (1.11), then the properties given by Eq. (1.9) can be deduced.

(a)

(b)

Figure 1.5 Sifting property of a delta function

## Example 3: The Inverse Fourier Transform

If we now approximate $\delta(t)$ by the function $N \operatorname{sinc}(N t)$ or $\delta(z-t)$ by $N \operatorname{sinc}(N(z-t))$, then the picture of the sifting property looks like Fig. 1.6. From Eq. (1.11), with this approximation $g(t) \approx g_{N}(t)$ where

$$
\begin{equation*}
g_{N}(t)=\int_{-\infty}^{\infty} N \operatorname{sinc}(\mathrm{~N}(z-t)) g(z) d z \tag{1.12}
\end{equation*}
$$

We now expect that as $N \rightarrow \infty, N \operatorname{sinc}(N(z-t)) \rightarrow \delta(z-t)$ and $g_{N}(t) \rightarrow g(t)$.


Figure 1.6 Using a sinc function to approximate a delta function

We can synthesize the function $N \operatorname{sinc}(N(z-t))$ from exponentials as shown in Eq. (1.7) in Example 2. Thus,

$$
\begin{equation*}
N \operatorname{sinc}(N(z-t))=\int_{-N / 2}^{N / 2} e^{-j 2 \pi(z-t) f} d f \tag{1.13}
\end{equation*}
$$

If we substitute Eq. (1.13) into Eq. (1.12) we can write

$$
\begin{equation*}
g_{N}(t)=\int_{-\infty}^{\infty} \int_{-N / 2}^{N / 2} e^{-j 2 \pi(z-t) f} d f g(z) d z \tag{1.14}
\end{equation*}
$$

Interchanging the order of integration in Eq. (1.14), we obtain

$$
\begin{equation*}
g_{N}(t)=\int_{-N / 2}^{N / 2} e^{j 2 \pi t f} \int_{-\infty}^{\infty} g(z) e^{-j 2 \pi z f} d z d f \tag{1.15}
\end{equation*}
$$

But in Eq. (1.15) the integral involving $g(z)$ is just the Fourier transform $G(f)$ from Eq. (1.1). Thus, Eq. (1.15) can be written as

$$
\begin{equation*}
g_{N}(t)=\int_{-N / 2}^{N / 2} G(f) e^{j 2 \pi t f} d f \tag{1.16}
\end{equation*}
$$

Now as $N \rightarrow \infty$ we have seen that we expect $g_{N}(t) \rightarrow g(t)$. Thus, Eq. (1.16) becomes

$$
\begin{equation*}
g(t)=\int_{-\infty}^{\infty} G(f) e^{j 2 \pi f t} d f \tag{1.17}
\end{equation*}
$$

which is just the inverse Fourier transform relation given by Eq. (1.3) that we set out to show.

## Example 4: Another Look at the Delta Function

Recall from Example 2 that as $N \rightarrow \infty$,

$$
N \operatorname{sinc}(N(z-t)) \rightarrow \delta(z-t)
$$

so that from Eq. (1.13) we can write

$$
\begin{equation*}
\delta(z-t)=\int_{-\infty}^{\infty} e^{-j 2 \pi(z-t) f} d f \tag{1.18}
\end{equation*}
$$

Eq. (1.18) is an important result expressing the delta function in terms of an exponential integral. Thus, although this integral does not exist in the ordinary sense, it can be written in terms of the delta function for purposes of calculation.

For example, with these results we can now verify Eq. (1.3) in a simple and straightforward way. Substituting Eq. (1.1) in Eq. (1.3) we can write

$$
\begin{align*}
g(t) & =\int_{-\infty}^{\infty} G(f) e^{j 2 \pi f t} d f \\
& =\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(z) e^{-j 2 \pi f z} d z e^{j 2 \pi f t} d f \tag{1.19}
\end{align*}
$$

Assume we can interchange the order of integration in Eq. (1.19) and then use Eq. (1.18) and Eq. (1.11) to write

$$
\begin{align*}
g(t) & =\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(z) e^{-j 2 \pi f(z-t)} d f d z \\
& =\int_{-\infty}^{\infty} g(z) \delta(z-t) d z=g(t) \tag{1.20}
\end{align*}
$$

thus verifying the inverse transform relation Eq. (1.3).
Finally, note that if $g(t)=1$ in Eq. (1.1), then

$$
\begin{equation*}
G(f)=\int_{-\infty}^{\infty} e^{-j 2 \pi f t} d t=\delta(f) \tag{1.21}
\end{equation*}
$$

We therefore obtain the Fourier transform pair

$$
\begin{equation*}
1 \Leftrightarrow \delta(f) \tag{1.22}
\end{equation*}
$$

This, of course, is the same as the result of Example 2 in the limit as $N \rightarrow \infty$.

### 1.2 Fourier Optics

In the Fourier transform $G(f)$ given by Eq. (1.1), the function $g(t)$ was thought of as a time signal where $t$ has the dimensions of time and the dimensions of $f$ are inverse time with units such as cycles/sec or Hertz $(\mathrm{Hz})$. It is also possible to take the Fourier transform of a spatial function $g(x)$, where $x$ has the units of length. In this case, we could write the Fourier transform as

$$
\begin{equation*}
G\left(v_{x}\right)=\int_{-\infty}^{\infty} g(x) e^{-j 2 \pi v_{x} x} d x \tag{1.23}
\end{equation*}
$$

where $\nu_{x}$ is a spatial frequency with units of inverse length, sometimes given as lines per millimeter.

We can define a two-dimensional Fourier transform, $G\left(v_{x}, v_{y}\right)$, of a twodimensional spatial function, $g(x, y)$, by the equation

$$
\begin{equation*}
G\left(v_{x}, v_{y}\right)=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j 2 \pi\left(v_{x} x+v_{y} y\right)} d x d y \tag{1.24}
\end{equation*}
$$

Consider the optical system shown in Fig. 1.7 in which a point source of coherent laser light at point S is focused by a lens at point T at the origin of the $p-q$ plane.


Figure 1.7 Fourier transforming property of a lens

If a glass plate containing an image with a transmittance given by $g(x, y)$ is placed in the $x-y$ plane in Fig. 1.7, then in Appendix A we show that the complex amplitude of the optical signal in the $p-q$ plane is given by

$$
\begin{equation*}
U(p, q)=K e^{j \frac{\pi}{\lambda D_{2}}\left(p^{2}+q^{2}\right)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j 2 \pi\left(v_{x} x+v_{y} y\right)} d x d y \tag{1.25}
\end{equation*}
$$

where

$$
\begin{equation*}
v_{x}=\frac{p}{\lambda D_{2}} \tag{1.26}
\end{equation*}
$$

and

$$
v_{y}=\frac{q}{\lambda D_{2}}
$$

and $\lambda$ is the wavelength of the laser light.
Thus, to within a complex constant, $U(p, q)$ is the two-dimensional Fourier transform of $g(x, y)$. Thus, we can write

$$
\begin{equation*}
U(p, q)=K e^{j \frac{\pi}{\lambda D_{2}}\left(p^{2}+q^{2}\right)} G\left(v_{x}, v_{y}\right) \tag{1.27}
\end{equation*}
$$

The light intensity in the $p-q$ plane is given by

$$
\begin{equation*}
I=|U(p, q)|^{2}=U(p, q) U^{*}(p, q)=\left|K G\left(v_{x}, v_{y}\right)\right|^{2} \tag{1.28}
\end{equation*}
$$

and is therefore proportional to the magnitude-squared of the two-dimensional Fourier transform of $g(x, y)$.

## Example 5: The Fourier Transform of a Slit

Consider the slit shown in Fig. 1.8a. If this slit is placed in the $x-y$ plane in Fig. 1.7, then the image produced in the $p-q$ plane will look like Fig. 1.8b.


Figure 1.8 (a) Slit in $x-y$ plane
(b) Fourier transform in $p$-q plane

The slit can be thought of as the product of two rect functions and we can write $g(x, y)$ as

$$
\begin{equation*}
g(x, y)=\operatorname{rect}\left(\frac{x}{N}\right) \operatorname{rect}\left(\frac{y}{M}\right) \tag{1.29}
\end{equation*}
$$

where $N$ is the width of the slit and $M$ is the height of the slit. From Eq. (1.8) we can write the Fourier transform of $g(x, y)$ in Eq. (1.29) as

$$
\begin{equation*}
G\left(v_{x}, v_{y}\right)=N \operatorname{sinc}\left(N v_{x}\right) M \operatorname{sinc}\left(M v_{y}\right) \tag{1.30}
\end{equation*}
$$

From Eq. (1.28), we see that the light intensity in the $p-q$ plane shown in Fig. 1.8b will be proportional to a $\sin c^{2}$ function of the type shown in Fig. 1.9.


Figure 1.9 The function $\operatorname{sinc}^{2}(f)$

In Fig. 1.8a, the height of the slit, $M$, is 25 times the width, $N$. From Eq. (1.30) it follows that the horizontal $\sin c^{2}$ function in the $p$-direction in Fig. 1.8b will be spread out 25 times wider than the vertical $\sin ^{2}$ function in the $q$ direction as shown. We will look more closely at scaling and shifting theorems in Chapter 2.

